The difference of two squares investigation

The relationship between the difference of two squares and the original numbers:

1) Difference of 1 (consecutive integers).

For example, the difference between 4^2 and 5^2 (which is 9).

Allow n to be the smaller value. The difference of the two squares is therefore:

$(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2$

$= 2n+1$

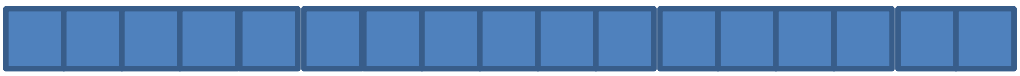
This is the general rule for any set of consecutive integers. What we see from this is that the difference of two consecutive squares **will always be an odd integer**.

Also, since there is no limit or constraint on what n can be, we can safely say that **any odd integer can be expressed as the product of two consecutive prime numbers** because 2n+1 defines every single odd integer. For example, we can find out that 95 is $48^2 - 47^2$ by setting it equal to 2n+1, and finding that n is 47.

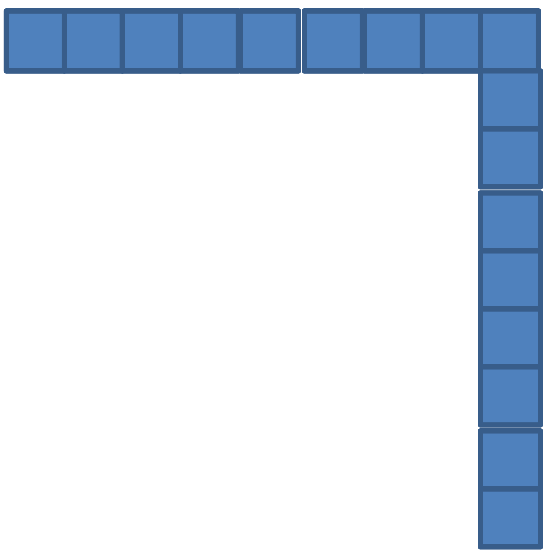
1 can be expressed as $1^2 - 0^2$.

A graphical proof of this idea is also possible using basic number theory:

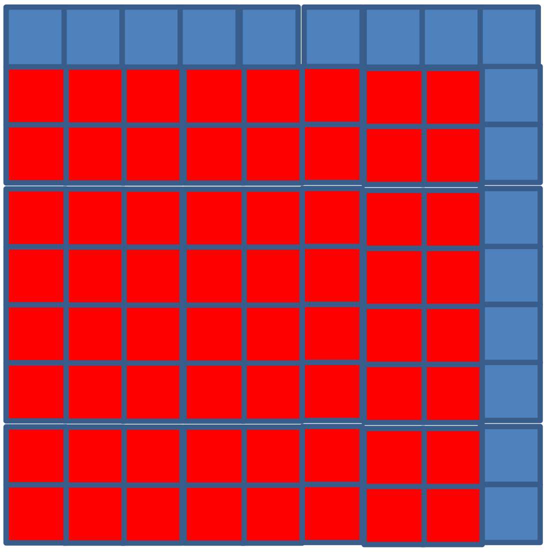
- Pick an odd number such as 17, and represent it with squares in a straight line:



- “Bend” it in half like so:



- Fill in the black space with more squares.



As you can see, the two squares (the smaller red one and the larger square which has a side length of one greater) have a difference of the blue squares – an odd number. Therefore, any odd number can be expressed as the difference of two consecutive squares.

2) Difference greater than 1

Let the difference between the two original numbers be p.

The difference of the two squares is therefore:

$(n+p)^2 - n^2 = n^2 + 2pn + p^2 - n^2$

$= p^2 + 2pn$

We can then factorise this expression to get p(p+2n).

We can see from this expression that the difference of two squares from any starting numbers will always be a multiple of their difference. This is because everything is multiplied by p.

For example, $55^2 - 50^2$ must be a multiple of 5 because there is a difference of 5 (55-50). The answer is 525, which does fit this rule.

When is the difference between two square numbers odd? And when is it even?

This can be determines using the following rules:

1) Even x Even = Even
2) Odd x Odd= Odd
3) Even – Even = Even **AND** Odd – Odd = Even
4) Even – Odd = Odd **AND** Odd – Even = Odd

We can represent the general difference of two squares as $A^2 – B^2$

Since a number squared is multiplying it by itself, Odd x Even is never possible.

We also see that the squared value is even if the original value is even, and vice versa (from rules 1 and 2)

Therefore, using rule 4, we can see that if one of A and B is odd, and the other even, the difference of two squares will be Odd. For example, if A = 10 and B = 5, 100 – 25 = 75. 75 is odd.

On the other hand, using rule 3, we see that if both A and B are either odd or even, the difference of two squares will be even. For example , if A = 11 and B = 5, 121 – 25 = 96. 96 is even.

Summary

- If A is even and B odd, or vice versa, the difference of two squares is odd.
- If both A and B are even or odd, the difference of two squares is even.

What do you notice about the numbers you CANNOT express as the difference of two perfect squares?

As seen from the first section of this document, all odd numbers can be expressed as the difference of two squares. But what about even numbers?

Well, as we have seen above, in order for this to be possible, A and B need to be both even or both odd, and therefore will have an even difference between them/

I wanted to return to the general equation of the difference of two squares:

$(a + b)^2 – a^2$ where b is the difference between the numbers, and a is the original, lower number.

This simplifies to $b^2 + 2ab$. As I said, in order to get an even result, the numbers must differ by an even amount. Therefore, b is a multiple of 2.

- From this, we can see that b^2 is a multiple of 4 because it is a multiple of 2 multiplied by itself.
- We also see that 2ab is a multiple of 4 regardless of what a is because a multiple of 2 (b) is multiplied by 2, making it a multiple of 4 overall.

Since b^2 and 2ab are both multiples of 4, their sum is also a multiple of 4.

Therefore, we see that the only even numbers that can be expressed as the difference of two squares are multiples of 4. For example, $16 = 4^2 – 0^2$, or $20 = 6^2 - 4^2$

OVERALL:

- Even numbers that are not multiples of 4 cannot be expressed as the difference of two squares, such as 6, 14 etc.
- Any other number can be.

Therefore, to answer the original question of "How many of the numbers from 1 to 30 can you express as the difference of two perfect squares?":

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25
26	27	28	29	30

Every number except 2,6,10,14,18,22,26,30

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