What's Possible - part 3: some proofs Zach Thompson

I was inspired by my mentor, John Slater who sent me an email from Amsterdam. He said I was **dangerously close** to being able to produce a formal proof for the fact that a set of consecutive positive odd integers starting at 1 always sum to a square number..... I like a challenge, so I had a think. Here goes:

<u>1) Gaussian Proof:</u>

Gauss famously summed consecutive numbers finding the formula for triangular numbers. I'm going to do the same, but sum consecutive odd numbers starting from 1. As $(2n-1)$ is the definition of an odd number, let $(2n-1)$ be the nth term.

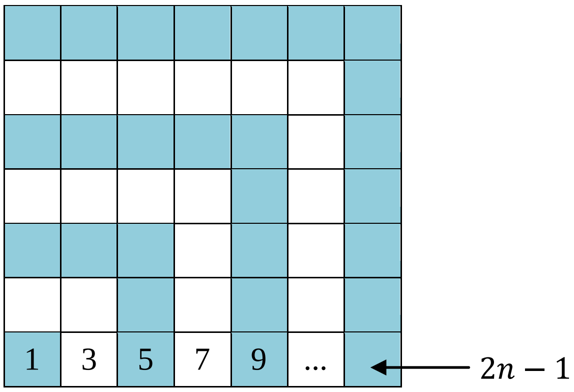
$$
\begin{array}{cccccccc}
 & 1 & 3 & 5 & \ldots & (2n-5) & (2n-3) & (2n-1) \\
+ & (2n-1) & (2n-3) & (2n-5) & \ldots & 5 & 3 & 1 \\
\hline
 & 2n & 2n & 2n & \ldots & 2n & 2n & 2n
\end{array}
$$

However, there are two strings of odd numbers. Therefore we must divide by 2.
Also there are n lots of $2n$, so we must multiply by n.
This leads to:

$$\frac{n \times 2n}{2} = n^2$$

This proves that summing consecutive odd numbers starting from 1 always sum to a square number.

<u>2) Geometric Proof:</u>



The numbers are arranged in a geometric pattern. To grow the sequence, start with a number (1), move it diagonally up to the right. Add 1 block to the left, and 1 block immediately below to complete the square. Continue forever. As the nth term is $2n-1$, the sides of the final square have length n. Each perfect square is made from the sum of consecutive odd numbers.

<u>3) By Induction (a bit grown up, but a clever shortcut):</u>

Start with a statement:

$$1+3+5+\cdots+(2n-3)+(2n-1)=n^2$$

And let the statement be true.

Suppose $n=1$, $\quad \therefore n^2=1$ and for the equation to balance, the left hand side MUST also equal 1.

So, if it is true for $n = 1$, is it also true for $(n + 1)$?

Substitute $n = (n + 1)$ into $(2n - 1)$ to get $(2(n + 1) - 1) = (2n + 2 - 1) = (2n + 1)$

$$1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1) + (2n + 1) = n^2 + (2n + 1)$$

BUT: $n^2 + 2n + 1$ can be factorised to $(n + 1)^2$, so yes, it does work for $(n + 1)$

Similarly:
Substitute $n = (n + 2)$ into $(2n - 1)$ to get $(2(n + 2) - 1) = (2n + 4 - 1) = (2n + 3)$

$$1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1) + (2n + 1) + (2n + 3) = n^2 + (2n + 1) + (2n + 3)$$

BUT: $n^2 + 2n + 1 + 2n + 3$ can be rewritten as : $n^2 + 4n + 4$ and factorised to $(n + 2)^2$
Again, it holds true for $(n + 2)$. We can continue like this forever.

Therefore summing consecutive odd numbers always sum to a square.

Final Thoughts (for the moment):
Since the digital root of any square number is 1, 4, 7, or 9, then any square number is either a multiple of 9, or 1 more than a multiple of 3. I could list many examples, here's one:

e.g. 529... DR = 7, 529 $-$1 = 528, 528÷3 = 176 (integer answer)

I think I've got closer, but I'm going to have a look at modular arithmetic next!!!!