This is what I have thought of:

If n is the side length of the chessboard, then every single chess board will have (at least) $n^2 + 1$ squares because n^2 refers the number of actual small squares, and the 1 refers to the whole square.

This suffices for 1x1 and 2x2, but clearly no more than that as 2x2 squares start to appear.

When you look at the 3x3 square, and spot all 4 of the 2x2 squares, you realise that the number of them is $(n-1)^2$: this is because the 2x2 squares require more room than a 1x1 square, and so you can fit (n-1) 2x2 squares on a side, and therefore $(n-1)^2$ in the entire square. So the formula for the total number of squares in a 3x3 square is $n^2 + 1 + (n-1)^{2,}$ which is 14.

The next thing that I noticed is that although the size of the chessboard increases, the largest possible square, apart from the whole thing itself, is always found 4 times. So although the quantity of the smaller squares, such as 2x2s in 5x5, increase, the 4x4s in a 5x5 is always 4. This means that we can rewrite the 3x3 equation as n^2 + 5, replacing the $(n-1)^2$ bit with 4

Unfortunately, as the sizes increase, the other smaller squares are not as cooperative! In 4x4s, although the number of 3x3s is always 4, the number of 2x2s still needs to be solved. This is where the $(n-1)^2$ part will be needed because, as I explained earlier, the 2x2s can always fit one less than the side length on one row/column. So for 4x4, the formula is n^2 + 5 + (n-1)^2 = 30

For 5x5s, we now have to deal with both 2x2s and 3x3s. Since 3x3s require more space than a 2x2, their quantity is determined by the formula $(n-2)^2$, making the overall count $n^2 + 5 + (n-1)^2 + (n-2)^2 =$ 55

This trend of adding on another $(n-x)^2$ as the square increases in size continues, such as the necessity to add on $(n-3)^2$ to the 6x6 square.

I noticed that the final bracket in which you subtract a value from n is always 3 less than n. Therefore, this is always 9, because $3^2 = 9$.

The second to last term always yields a 16 value because the result of n - x is always 4.

So what is special here? They are all square numbers!

For 4x4, I gave the general formula as $n^2 + 5 + (n-1)^2 = 30$.

For 5x5, I gave the general formula as $n^2 + 5 + (n-1)^2 + (n-2)^2 = 55$.

It can be observed that as the size of the square increases, the number of $(n-x)^2$ terms also increases because there are more types of small squares to consider.

But how to write as a general formula? We know that we must keep the n^2+ 5 bit because it is present in every square. To try and decipher this, I expanded the bracket of the $(n-1)^2 + (n-2)^2$ ... bits for each square and found the following results:

4x4: $n^2 - 2n + 1$

5x5: $2n^2 - 6n + 5$

6x6: $3n^2 - 12n + 14$

7x7: $4n^2 - 20n + 30$

8x8 : $5n^2 – 30n + 55$

For each part, I then need to find the nth term of the sequences, but replace n with n-3 because I am starting with the side length 4 as the first term.

For the n^2 part, the coefficient clearly increases by 1 each time, so the normal nth term would be $(n)(n^2)$ and so in relation to the side length n, it is $(n-3)(n^2)$. Therefore, so far, we have got $n^2 + 5 + (n-3)(n^2)$ as our general formula.

For the n part, the coefficient increases by an increasingly large amount each time: the 2nd difference becomes 2. The normal nth term would therefore be n(n+1) and so in relation to side length n, it is (n-3)(n-2) n, . Therefore, so far, we have got $n^2 + 5 + (n-3)(n^2) - (n-3)(n-2)n$ as our general formula.

Finally, for the regular number, it adds on a new square number that increases each time, such as 2^2 from a 4x4 to a 5x5, but 9 from a 5x5 to a 6x6. The nth term is therefore: $(n-3)^3/3 + (n-3)^2/2 + (n-3)/6$.

I believe that it is safe to assume that all further square sizes increase in the same way: the x^2 part will always increase by 1 because you are adding on a new $(n-x)^2$ part each time, the sequence will continue endlessly for the n coefficient because the x part in $(n-x)^2$ will continue to increase by 1 (and therefore increase the n coefficient by 2), and finally, the square numbers will continue to be added on as $(n-x)^2$ will always lead to an x^2 value, which must therefore be a square number.

Thus the final general formula: $n^2 + 5 + (n-3)(n^2) - (n-3)(n-2)n + (n-3)^3/3 + (n-3)^2/2 + (n-3)/6$

Therefore, for an 8x8 board, the number of squares is:

64 + 5 + 320-240 + 125/3 + 25/2 + 5/6 = 204

Of course, since this is a general formula, we can replace n with any integer value to find the number of squares in it, for example, a 100x100 square would have a total of 338350 squares in.

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