

Assume that we go up the thirteen steps only by 1 step, and that we do not take steps two-at-a-time.

This is equivalent to asking how many ways are there to order a sequence of 13 ones.

There are only ${}_{13}C_0$, or only 1 way to go up the steps.

Now, assume that we go up the thirteen steps, and this time we only have one step that is two-at-a-time.

This is equivalent to asking how many ways are there to order a sequence of 11 ones and 1 two.

There are ${}_{12}C_1$, or 12 ways to get up the steps.

We continue with going up the steps, two of the steps being two-at-a-time.

This is equivalent to asking how many ways are there to order a sequence of 9 ones and 2 twos.

There are ${}_{11}C_2$, or 55 ways to get up the steps.

Following the same logic,

For three of the steps being two-at-a-time, there are ${}_{10}C_3$, or 120 ways to get up the steps.

For four of the steps being two-at-a-time, there are ${}_{9}C_4$, or 126 ways to get up the steps.

For five of the steps being two-at-a-time, there are ${}_{8}C_5$, or 56 ways to get up the steps.

For six of the steps being two-at-a-time, there are ${}_{7}C_6$, or 7 ways to get up the steps.

And there are no more cases available.

Thus, the total number of ways is $1 + 12 + 55 + 120 + 126 + 56 + 7$, or 377 ways in total.

Generalising the Result:

We could express the number of ways to get up the steps for each case based on the number of two-at-a-time steps, which we will call α , and the total number of steps, which we will call β : $\beta - \alpha C_\alpha$. α and β are both integers in this case.

By definition of a combination, the number to the left, in this case $\beta - \alpha$, has to be equal or greater than the number on the right, or in this case, α : $\beta - \alpha \geq \alpha$. Adding α to both sides and dividing by 2, we get $\alpha \leq \beta/2$. Thus, the number of two-at-a-time steps possible in a series of β steps is $0 \leq \alpha \leq \beta/2$, but since α is always an integer, we can use the floor function on the inequality, $0 \leq \alpha \leq \lfloor \beta/2 \rfloor$.

Now, the total number of ways to get up the steps is just the sum of the number of ways for each case, so we calculate the following summation:

$$\sum_{\alpha=0}^{\lfloor \beta/2 \rfloor} \beta - \alpha C_\alpha = F(\beta+1)$$

The summation, for any total number of steps β greater than zero, is equivalent to the $(\beta+1)$ th Fibonacci number $F(\beta+1)$, the proof of which will not be shown here, but it can be proved using induction and the theorem ${}_nC_m + {}_nC_{m+1} = {}_{n+1}C_{m+1}$ or Zeckendorf's theorem. One can calculate the Fibonacci number $F(n)$ using Binet's formula:

$$\frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}},$$