Because the difference between two consecutive squares is odd, then all the odd numbers are the difference of perfect squares. Therefore all the odd numbers from 1 to 30 can be expressed as the difference of two perfect squares.

Proof:
Consecutive numbers can be defined as: $n,\ (n+1),\ (n+2),\ (n+3)$ etc... where n is a whole number (this is because 0 is allowed in this question).

Therefore the difference of two consecutive squares = $(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = (2n+1)$
And $(2n+1)$ is the definition of any odd number, as $2n$ is always an even number because it is a multiple of 2.

Therefore we can calculate:

$1 = (1^2 - 0^2) = 1 - 0$
$3 = (2^2 - 1^2) = 4 - 1$
$5 = (3^2 - 2^2) = 9 - 4$
$7 = (4^2 - 3^2) = 16 - 9$
$9 = (5^2 - 4^2) = 25 - 16$
$11 = (6^2 - 5^2) = 36 - 25$
$13 = (7^2 - 6^2) = 49 - 36$
$15 = (8^2 - 7^2) = 64 - 49$
$17 = (9^2 - 8^2) = 81 - 64$
$19 = (10^2 - 9^2) = 100 - 81$
$21 = (11^2 - 10^2) = 121 - 100$
$23 = (12^2 - 11^2) = 144 - 121$
$25 = (13^2 - 12^2) = 169 - 144$
$27 = (14^2 - 13^2) = 196 - 169$
$29 = (15^2 - 14^2) = 225 - 196$

To try and find the even numbers from 1 to 30, I investigated the same pattern but with a difference of 2:
so $(n+2)^2 - n^2 = n^2 + 4n + 4 - n^2 = 4n + 4 = 4(n+1)$ This means that multiples of 4 can be expressed as the difference of two perfect squares where the squares differ by 2:

$4 = (2^2 - 0^2) = 4 - 0$
$8 = (3^2 - 1^2) = 9 - 1$
$12 = (4^2 - 2^2) = 16 - 4$
$16 = (5^2 - 3^2) = 25 - 9$
$20 = (6^2 - 4^2) = 36 - 16$
$24 = (7^2 - 5^2) = 49 - 25$
$28 = (8^2 - 6^2) = 64 - 36$

Similarly, I then looked at squares with a difference of 3:
so $(n+3)^2 - n^2 = n^2 + 6n + 9 - n^2 = 6n + 9 = 3(2n+3)$

The answer will always be odd because inside the bracket you have an even number $(2n)$ plus an odd number (3), which will always sum to an odd number. Because this is then multiplied by 3, this means that odd multiples of 3 can be expressed as the difference of two perfect squares where the squares differ by 3. Using whole numbers $3^2 - 0^2 = 9$ is the smallest odd multiple of 3 which works using this method:

$9 = (3^2 - 0^2) = 9 - 0$
$15 = (4^2 - 1^2) = 16 - 1$
$21 = (5^2 - 2^2) = 25 - 4$
$27 = (6^2 - 3^2) = 36 - 9$

At this point, I realised that an even result could only be produced by both square numbers being, both even or both odd, because $\mathrm{O} - \mathrm{O} = \mathrm{E},\ \mathrm{E} - \mathrm{E} = \mathrm{E},\ \mathrm{O} - \mathrm{E} = \mathrm{O}$ and $\mathrm{E} - \mathrm{O} = \mathrm{O}$

Therefore, I will only try square numbers with an even integer difference:

$$(n + \text{even positive integer})^2 - n^2$$

So I tried: $(n+4)^2 - n^2 \;=\; n^2 + 8n + 16 - n^2 = \; 8n + 16 = \; 8(n+2)$
8 is divisible by 4, and we've already found that all multiples of 4 can be expressed as the difference of two perfect squares. Similarly:

$$(n+6)^2 - n^2 = 12(n+3)$$
$$(n+8)^2 - n^2 = 16(n+4)$$
$$(n+10)^2 - n^2 = 20(n+5) \qquad \text{etc...}$$

This proves that multiples of 4 work, but I am looking for even numbers which are NOT multiples of 4:

2, 6, 10, 14, 18, 22, 26, and 30

To try to find these numbers I adjusted my basic formula:$(n + \text{positive integer})^2 - n^2$ as follows:

Let the integer be $4k-2$, where k is a positive integer, and $4k$ is a multiple of 4. This gives us:
$(n+4k-2)^2 - n^2 \;=\; 8kn - 4n + 16k^2 - 16k + 4 = \; 4(2kn - n + 4k^2 - 4k + 1)$

Similarly let the integer be $4k$, where k is a positive integer, and $4k$ is a multiple of 4. This gives us:
$(n+4k)^2 - n^2 \;=\; n^2 - 8kn + 16k^2 - n^2 = \; 8kn + 16k^2 = \; 4(2kn + 4k^2)$

This means that multiples of 4 can be expressed as the difference of two perfect squares where the squares differ by either any even number which is not divisible by 4, or any multiple of 4. All even numbers fall into one of these classes, because all even numbers are divisible by 2, but only half can be further divided by 2. Therefore the difference of two perfect squares, where the squares differ by any even number will always be a multiple of 4.

Similarly we can investigate square numbers with an odd integer difference:

$$(n + \text{odd positive integer})^2 - n^2$$

Let the odd positive integer be $2k+1$, since that is the definition of an odd number. This gives us:
$(n+2k+1)^2 - n^2 = \; 4kn + 2n + 4k^2 + 4k + 1$, which can be simplified to:

$$2(2kn + n + 2k^2 + 2k) + 1$$

The first part of the expression is divisible by 2, and therefore even: $2(2kn + n + 2k^2 + 2k)$, but the addition of 1, at the end, makes the expression odd.

Therefore we can write a general rule for the difference of two squares , where n is any positive integer:

$$(n + \text{even positive integer})^2 - n^2 = \text{even positive integer which is divisible by 4 and not otherwise}$$

$$(n + \text{odd positive integer})^2 - n^2 = \text{odd positive integer}$$

<u>Conclusion:</u>
There are 22 integers from 1 to 30 which can be expressed as the difference of two perfect squares: 1, 3, 4, 5, 7, 8, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21, 23, 24, 25, 27, 28, 29

But 2, 6, 10, 14, 18, 22, 26 and 30 cannot be expressed as the difference of two perfect squares.

This method does prove that this is the only solution.