I've been thinking about a problem that my mentor John Slater from UKMT, asked me to solve. I'm interested in modulo and sets of numbers, so I think that's why he asked me. Although he didn't give me any clues, it wasn't that hard really, but great fun to play with and made me think about how numbers are structured.

As I worked to answer John's question, I had a brainwave, and started to ask my own questions, this is pretty usual for me. I took John's problem and extended it, because I wanted to look at it in a different way.

What I found was really interesting and very exciting. The patterns are beautiful, the more I did the more fantastic it became, and by the end my heart was racing!!

I've invented a sieve for primes, using modulo. This is my hypothesis. Even though it's brilliantly simple, I've ended up with more questions that I started with, which is awesome!!! I want to find the answers, I want proof.

Because there's a lot of stuff here, John has suggested I organise my work into chapters. Here goes:

Chapter 1: John's Question, part (a)

Look at the following sequence of numbers: 1...5...9...13...17...21...25...29...33...37...41...45...49....
What is the smallest number that can be a multiple of two of these numbers in two different ways that also appear in this sequence?

This was quite straightforward, and didn't take that long. This was my answer:

These numbers are one more than a multiple of 4. They can all be expressed as $4n+1$ and/or 1(mod4).

If these numbers were the only numbers that existed, only those that are products of 1 and itself are s-prime. This follows the rules of primes except instead of natural numbers, all numbers are 1(mod 4).

The LCM of 5,9 = 45, but 45 but doesn't meet the criteria as 5×9 is a reversal of 9×5. As all the numbers in the sequence are 1 more than a multiple of 4, the difference between any two of these numbers must be a multiple of 4.

Therefore the answer is $(45 \times 4) + 45 = 45 \times 5 = 225$, which can be expressed as 5×45 and 9×25, all of which are congruent to 1(mod 4).

1 isn't allowed as a factor, so 225 is the lowest example. (25 doesn't work as it's factors are 1×25 and 5×5).

All of this works because 1(mod 4) × 1(mod 4) = 1(mod 4)

The rule is:

$$a \equiv 1(mod4) \times b \equiv 1(mod4) = ab \equiv 1(mod\ 4)$$

$$ab \equiv 1(mod\ 4) \times 5 \equiv 1(mod4) = 5ab \equiv 1(mod\ 4)$$

The smallest value of a and b are 5 and 9, producing 225. The next pair is 5 and 13, which produce 325. This can be factored as 5×65 and 13×25. This continues.... forever. For example, the next five numbers are:

9 and 9 produce 405, which is: 5×81 and 9×45,
5 and 17 produce 425, which is: 5×85 and 17×25,
5 and 21 produce 525, which is: 5×105 and 21×25,
9 and 13 produce 585, which is: 9×65 and 13×45,
5 and 29 produce 725, which is: 5×145 and 25×29, etc...

A number in the real world of all natural numbers is either prime or composite. However a number in the sequence is either a s-prime or a multiple of s-primes. But a prime is not the same as a s-prime in some cases, e.g. 9 is a s-prime because 3 is not in the sequence and 9 is not divisible by 5. But 9 is composite in the real world.

The numbers I have found so far are s-primes × s-composite. The way I created these numbers (the rule) means that this will always happen.

Chapter 2: Unique Factorisation, John's Question part (b)

Because I'm obsessed with primes, this was an interesting thought. Could I find an exception? Again, this wasn't tough, I just needed to get inside the numbers. This is what I did:

To find a number which is an s-prime × s-prime, list all of the s-primes:
5...9...13...17...21...29...33...37...41...49...53...57...etc...

Multiply two of the s-primes until you get to a number whose factors can be arranged in two different ways.
I started to calculate, but it would take too long, so I used a multiplication table instead:

×	5	9	13	17	21	29	33	37	41	49
5	25	45	65	85	105	145	165	185	205	245
9	45	81	117	153	189	261	297	333	369	441
13	65	117	169	221	273	377	429	481	533	637
17	85	153	221	289	357	493	561	629	697	833
21	105	189	273	357	441	609	693	777	861	1029
29	145	261	377	493	609	841	957	1073	1189	1421
33	165	297	429	561	693	957	1089	1221	1353	1617
37	185	333	481	629	777	1073	1221	1369	1517	1813
41	205	369	533	697	861	1189	1353	1517	1681	2009
49	245	441	637	833	1029	1421	1617	1813	2009	2401

This method proves that 441 is the smallest s-composite that is the product of two s-primes in two different ways.

This shows that unique factorisation fails in this sequence. 441 is the first number to show this.

<u>The Big Question</u>: if the Fundamental Theorem of Arithmetic fails for this sequence, does this mean it will fail for sequences in other moduli?

<u>Other Observations:</u> when I was looking at the s-primes, I noticed that there were some square numbers in there. These are of the form $(4n+3)^2$ or $[3(\text{mod } 4)]^2$, because $[3(\text{mod } 4)]^2 = 9(\text{mod } 4) \equiv 1(\text{mod } 4)$

Chapter 3: 'Rotten Numbers' in 1(mod 4)

Even before I answered John's question, I wanted to see what would happen if I only retain numbers where **all** of the divisors of any number used, are congruent to 1(mod 4).

My Question, a Challenge to Myself:

I wanted to cast out 'rotten numbers', which appear prime (in the sequence), but are not in the real world. That is why I called them 'rotten', because they look good on the surface, but are not 'pure' inside. This became really, really interesting and created amazingly beautiful patterns.

To ensure that all of the divisors of any number used also appears in the sequence, get rid of any 'rotten numbers'. A 'rotten number' is a number which has at least one factor that is not congruent to 1(mod 4).

Here is a list of all numbers congruent to 1(mod 4):
1...5...9...13...17...21...25...29...33...37...41...45...49....53...57...etc...

My Thoughts:

Observations: Every third number is a multiple of three, and 'rotten' as a result. Square numbers in the form $(4n+3)^2$ or $[3(\text{mod } 4)]^2$ are 'rotten'. All multiples of the square root of $(4n+3)^2$ or $[3(\text{mod } 4)]^2$ are 'rotten', e.g. $77 = \sqrt{49} \times \sqrt{121}$ or $77 = \sqrt{49} \times 11$ or $77 = \sqrt{121} \times 7$

NOTE: DON'T SIMPLIFY!!!!* It's really important to keep the expression in this form!!!!

The numbers that are left are:
List of all the numbers which aren't 'rotten' are:
1...5...13...17...25...29...37...41...53...61...65...73...85...89...97...101...109...113...125...137...145...149...157...169 ...173...181...185...193...197...205...221...229...233...241...257...265...269...277...281...289...293...305...313... 317...325...337...349...353...365...373...377...389...397...etc...

From this list, the s-primes are:
5...13...17...29...37...41...53...61...73...89...97...101...109...113...137...149...157...173...181...193...197...229... 233...241...257...269...277...281...293...313...317...337...349...353...373...389...397...

And from this list, the s-composites are:
25...65...85...125...145...169...185...205...221...265...289...305...325...365...377...

Following my rule for s-prime × s-composite, it's easy to find the first few numbers:

$(5 \times 13) \times 5 = 325 \equiv 1(\text{mod } 4) = 5 \times 65$ or 25×13
$(5 \times 17) \times 5 = 425 \equiv 1(\text{mod } 4) = 5 \times 85$ or 25×17
$(5 \times 25) \times 5 = 625 \equiv 1(\text{mod } 4) = 5 \times 125$ or 25×25
$(5 \times 29) \times 5 = 725 \equiv 1(\text{mod } 4) = 5 \times 145$ or 25×29
$(5 \times 37) \times 5 = 925 \equiv 1(\text{mod } 4) = 5 \times 185$ or 25×37
$(5 \times 41) \times 5 = 1025 \equiv 1(\text{mod } 4) = 5 \times 205$ or 25×41
$(13 \times 17) \times 5 = 1105 \equiv 1(\text{mod } 4) = 13 \times 85$ or 65×17...

I realised that I missed 5 and 25, which produce 625 from John's question. Oooops!

But to find numbers that are just s-prime × s-prime, is impossible as my s-prime list has shrunk to a real prime list.

My Predictions:

I predict that this will continue, forever and I think this could be used to predict prime numbers that are congruent to 1(mod 4). This is because all s-primes, that are not real primes become 'rotten', as they are divisible by a number which is not in the sequence (and congruent to 3(mod 4)).

This is really exciting!!

I wonder if the same would occur in other moduli?

My Models:

But first, I'm going to plot the s-primes congruent to 1(mod 4) that have been stripped of any 'rotten numbers.

(Please see the array's on the next page)

PINK = rotten numbers - remove these first

BLUE = s-composites left over after rotten numbers have been stripped out

YELLOW = s-primes congruent to 1(mod 4), stripped of 'rotten' numbers/s-composites

1	5	9	13	17	21
25	29	33	37	41	45
49	53	57	61	65	69
73	77	81	85	89	93
97	101	105	109	113	117
121	125	129	133	137	141
145	149	153	157	161	165
169	173	177	181	185	189
193	197	201	205	209	213
217	221	225	229	233	237
241	245	249	253	257	261
265	269	273	277	281	285
289	293	297	301	305	309
313	317	321	325	329	333
337	341	345	349	353	357
361	365	369	373	377	381
385	389	393	397	401	405
409	413	417	421	425	429
433	437	441	445	449	453
457	461	465	469	473	477
481	485	489	493	497	501
505	509	513	517	521	525
529	533	537	541	545	549
553	557	561	565	569	573
577	581	585	589	593	597
601	605	609	613	617	621
625	629	633	637	641	645
649	653	657	661	665	669
673	677	681	685	689	693
697	701	705	709	713	717
721	725	729	733	737	741
745	749	753	757	761	765
769	773	777	781	785	789
793	797	801	805	809	813
817	821	825	829	833	837
841	845	849	853	857	861
865	869	873	877	881	885
889	893	897	901	905	909
913	917	921	925	929	933
937	941	945	949	953	957
961	965	969	973	977	981
985	989	993	997	1001	1005
1009	1013	1017	1021	1025	1029
1033	1037	1041	1045	1049	1053
1057	1061	1065	1069	1073	1077
1081	1085	1089	1093	1097	1101
1105	1109	1113	1117	1121	1125
1129	1133	1137	1141	1145	1149
1153	1157	1161	1165	1169	1173

1	5	9	13	17
21	25	29	33	37
41	45	49	53	57
61	65	69	73	77
81	85	89	93	97
101	105	109	113	117
121	125	129	133	137
141	145	149	153	157
161	165	169	173	177
181	185	189	193	197
201	205	209	213	217
221	225	229	233	237
241	245	249	253	257
261	265	269	273	277
281	285	289	293	297
301	305	309	313	317
321	325	329	333	337
341	345	349	353	357
361	365	369	373	377
381	385	389	393	397
401	405	409	413	417
421	425	429	433	437
441	445	449	453	457
461	465	469	473	477
481	485	489	493	497
501	505	509	513	517
521	525	529	533	537
541	545	549	553	557
561	565	569	573	577
581	585	589	593	597
601	605	609	613	617
621	625	629	633	637
641	645	649	653	657
661	665	669	673	677
681	685	689	693	697
701	705	709	713	717
721	725	729	733	737
741	745	749	753	757
761	765	769	773	777
781	785	789	793	797
801	805	809	813	817
821	825	829	833	837
841	845	849	853	857
861	865	869	873	877
881	885	889	893	897
901	905	909	913	917
921	925	929	933	937
941	945	949	953	957
961	965	969	973	977

1	5	9	13
17	21	25	29
33	37	41	45
49	53	57	61
65	69	73	77
81	85	89	93
97	101	105	109
113	117	121	125
129	133	137	141
145	149	153	157
161	165	169	173
177	181	185	189
193	197	201	205
209	213	217	221
225	229	233	237
241	245	249	253
257	261	265	269
273	277	281	285
289	293	297	301
305	309	313	317
321	325	329	333
337	341	345	349
353	357	361	365
369	373	377	381
385	389	393	397
401	405	409	413
417	421	425	429
433	437	441	445
449	453	457	461
465	469	473	477
481	485	489	493
497	501	505	509
513	517	521	525
529	533	537	541
545	549	553	557
561	565	569	573
577	581	585	589
593	597	601	605
609	613	617	621
625	629	633	637
641	645	649	653
657	661	665	669
673	677	681	685
689	693	697	701
705	709	713	717
721	725	729	733
737	741	745	749
753	757	761	765
769	773	777	781

My Observations:
I've used different array sizes to show up patterns.

In the 6 column array, the 3rd and 6th columns are 'rotten', as they are divisible by 3.

In the 5 column array, the 2nd column, apart from 5, entirely consists of 'rotten' numbers and s-composites.

In the 4 column array, the multiples of 3 form a z-pattern. Follow the diagonal 9, 21, 33, then straight across to get the next diagonal which is 45, 57, 69, and 81. This continues forever.

Each of the numbers in yellow are s-primes congruent to 1(mod4), stripped of 'rotten numbers'/s-composites, and they are all primes in the real world; but this is not a complete list. It is approximately 48%. This comes from the 6 column table above, which counts primes all the way up to 1153 which is the 192nd (real) prime number. Of these 92 primes are in the sequence 1(mod 4), so yellow. Therefore the ratio of s-primes to real primes in this range is 47.916 recurring, or approximately 48%

I do expect the ratio to be 50/50, because I have a feeling that half the primes will be congruent to 1(mod 4) and the other half congruent to 3(mod 4). So I thought some more.

Next Steps:
I've previously proved that except for 2 and 3, all real primes can be expressed as $6n \pm 1$, which is the same as 1(mod6) or 5(mod 6). If I arrange my results over a 6 column array, but this time use all of the natural numbers, hopefully any patterns should be clearer. Obviously not all the prime numbers will be identified, only those congruent to 1(mod 4). Similarly, not all the non-primes will be eliminated, only those congruent to 1(mod 4).

1 is neither composite nor prime, so can be eliminated. All multiples of 2 and 3, except for 2 and 3, can also be eliminated. These numbers are all grey.

The chart is on the next page (please have a look before reading any more).

Map for primes congruent to 1(mod 4)

a) PINK = rotten numbers - remove these first, following my rules
b) BLUE = s-composites left over after rotten numbers have been stripped out
c) YELLOW = s-primes congruent to 1(mod 4), stripped of 'rotten' numbers/s-composites
d) GREY = 1, multiples of 2 except 2 itself, and multiples of 3 that are not congruent to 1(mod4) except 3 itself

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60
61	62	63	64	65	66
67	68	69	70	71	72
73	74	75	76	77	78
79	80	81	82	83	84
85	86	87	88	89	90
91	92	93	94	95	96
97	98	99	100	101	102
103	104	105	106	107	108
109	110	111	112	113	114
115	116	117	118	119	120
121	122	123	124	125	126
127	128	129	130	131	132
133	134	135	136	137	138
139	140	141	142	143	144
145	146	147	148	149	150
151	152	153	154	155	156
157	158	159	160	161	162
163	164	165	166	167	168
169	170	171	172	173	174
175	176	177	178	179	180
181	182	183	184	185	186
187	188	189	190	191	192
193	194	195	196	197	198
199	200	201	202	203	204
205	206	207	208	209	210
211	212	213	214	215	216
217	218	219	220	221	222
223	224	225	226	227	228
229	230	231	232	233	234
235	236	237	238	239	240
241	242	243	244	245	246
247	248	249	250	251	252
253	254	255	256	257	258
259	260	261	262	263	264

265	266	267	268	269	270
271	272	273	274	275	276
277	278	279	280	281	282
283	284	285	286	287	288
289	290	291	292	293	294
295	296	297	298	299	300
301	302	303	304	305	306
307	308	309	310	311	312
313	314	315	316	317	318
319	320	321	322	323	324
325	326	327	328	329	330
331	332	333	334	335	336
337	338	339	340	341	342
343	344	345	346	347	348
349	350	351	352	353	354
355	356	357	358	359	360
361	362	363	364	365	366
367	368	369	370	371	372
373	374	375	376	377	378
379	380	381	382	383	384
385	386	387	388	389	390
391	392	393	394	395	396
397	398	399	400	401	402
403	404	405	406	407	408
409	410	411	412	413	414
415	416	417	418	419	420
421	422	423	424	425	426
427	428	429	430	431	432
433	434	435	436	437	438
439	440	441	442	443	444
445	446	447	448	449	450
451	452	453	454	455	456
457	457	459	460	461	462
463	464	465	466	467	468
469	470	471	472	473	474
475	476	477	478	479	480
481	482	483	484	485	486
487	488	489	490	491	492
493	494	495	496	497	498
499	500	501	502	503	504
505	506	507	508	509	510
511	512	513	514	515	516
517	518	519	520	521	522
523	524	525	526	527	528

My Conclusions for Real Primes in 1(mod 4):

It is very remarkable that the numbers appear to follow a regular pattern. The lines that contain real primes congruent to 1(mod 4) appear to alternate with those that don't. Usually the gap is only 1 line, but occasionally there is a 'little bump', and the gap can increase to 3, 5, or 7, based on the numbers up to 528 (this could all change). It is also interesting that 3, 5, and 7 are themselves prime numbers.

I seem to have devised a sieve, which shows up primes congruent to 1(mod 4) and may explain the gaps between these numbers.

For example: take 197 (a real prime):

(A) The next line has 199 and 203, which are not congruent to 1(mod 4). Therefore they are left blank.

(B) The next line has 205 and 209, which are congruent to 1(mod 4), BUT are either s-composite or 'rotten' (divisible by a number which is not congruent to 1(mod 4)).

Similarly, the next line follows pattern A, and the following line pattern B. Finally, the next line follows pattern A, and the following line contains real primes 229 and 233. This explains the gap between 197 and 229. There is a similar argument to explain the gap between 461 and 509, etc...

Chapter 4: 'Rotten Numbers' in 3(mod 4)

I need to use other moduli to explain the blank lines and find the rest of the primes.

The remaining numbers are:
7...11...19...23...31...35...43...47...55...59...67...71...79...83...91...95...103...107...115...119...127...131...139...143

There seems to be a regular interval of 4, 8, 4, 8,... between the numbers, which made me think.

These numbers are all congruent to 3(mod 4), which is the other half of the real odd numbers. So in this case, the modulo is the same (mod 4).

Important note!! New Definition for 'Rotten Numbers' in 3(mod4): T-Primes are Born:

To ensure that all of the divisors of any number used also appears in the sequence, once again I have to get rid of any 'rotten numbers'. But this time, a 'rotten' number is a number which has at least one factor that is not congruent to 3(mod 4) EXCEPT for 1 itself.

This is important as the only way to express a composite in this sequence is 1(mod4) × 3(mod 4) = 3(mod 4). Therefore all composites are 'rotten' in this sequence, and only the real primes remain. This is because only real primes have 1 as a factor congruent to 1(mod 4), in this sequence.

Therefore a 'rotten' number in this sequence is of the form:
1(mod 4) × 3(mod 4) , where 1(mod 4) > 1

In the 1(mod 4) sequence, the numbers were labelled s-primes and s-composites. In this 3(mod 4) sequence I will call them T-primes and T-composites.

Examples:
E.g. 7 is a T-prime and a real prime because it's only factors are 1 and 7, 7≡ 3(mod 4) and 1 cannot be greater than 1. 35 is a T-composite because it's factors are 5 and 7, where 5≡1(mod4) and 7≡ 3(mod 4), and 5> 1, so 'rotten'.

Therefore any number in this sequence with a real prime factor congruent to 1(mod 4) must be 'rotten', as 1 is not prime, the real prime factor must be >1, and therefore cannot appear in the sequence 3(mod 4).

It's really easy now, use my sieve to find the real primes from the 1(mod 4) sequence, and then eliminate multiples of these real primes in the 3(mod 4) sequence.

What a beautiful cycle!!!!!!

E.g. 5, 13, 17, 29, 37, 41, 53, 61 etc... were identified as real primes congruent to 1(mod 4). Each of these numbers are greater than 1. Therefore any multiples of these numbers that appear in the 3(mod 4) sequence must be cast out as 'rotten', according to my rule.

Test:
The 'blank' numbers are:

7...11...19...23...31...35...43...47...55...59...67...71...79...83...91...95...103...107...115...119...127...131...139...143...151...155...163...167...175...179...187...191...199...203...211...215...223...227...235...239...247...251...259...263...271...275...283...287...295...299...307...311...319...323...331...335...343...347...355...359...367...371...379...383...391...395...403...407...415...419...427...431...439...443...451...455...463...467...475...479...487...491...499...503...511...515...523...527...etc...

The list of all the numbers which aren't 'rotten' are:

7...11...19...23...31...43...47...59...67...71...79...83...103...107...127...131...139...151...163...167...179...191...199...211...223...227...239...251...263...271...283...307...311...331...347...359...367...379...383...419...431...439...443...463...467...479...487...491...499...503...523...etc...

ALL of these numbers are T-primes and real primes. My method eliminated all of the T-composites.

More Models:
Again, I plotted my results. (see chart on the next page).

343 was interesting because it is $7 \times 7 \times 7$ but also 7×49.
49 is not in the sequence, therefore 343 is 'rotten'.

Map for primes congruent to 1(mod 4) + primes congruent to 3(mod 4) = MAP FOR ALL PRIMES

e) USE the map for primes congruent to 1(mod 4) as a base
f) GREEN = rotten numbers - remove these first, following my rules
g) ORANGE = T-primes congruent to 3(mod 4), stripped of 'rotten' numbers
h) BLANK = the real primes 2 and 3

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60
61	62	63	64	65	66
67	68	69	70	71	72
73	74	75	76	77	78
79	80	81	82	83	84
85	86	87	88	89	90
91	92	93	94	95	96
97	98	99	100	101	102
103	104	105	106	107	108
109	110	111	112	113	114
115	116	117	118	119	120
121	122	123	124	125	126
127	128	129	130	131	132
133	134	135	136	137	138
139	140	141	142	143	144
145	146	147	148	149	150
151	152	153	154	155	156
157	158	159	160	161	162
163	164	165	166	167	168
169	170	171	172	173	174
175	176	177	178	179	180
181	182	183	184	185	186
187	188	189	190	191	192
193	194	195	196	197	198
199	200	201	202	203	204
205	206	207	208	209	210
211	212	213	214	215	216
217	218	219	220	221	222
223	224	225	226	227	228
229	230	231	232	233	234
235	236	237	238	239	240
241	242	243	244	245	246
247	248	249	250	251	252
253	254	255	256	257	258
259	260	261	262	263	264

265	266	267	268	269	270
271	272	273	274	275	276
277	278	279	280	281	282
283	284	285	286	287	288
289	290	291	292	293	294
295	296	297	298	299	300
301	302	303	304	305	306
307	308	309	310	311	312
313	314	315	316	317	318
319	320	321	322	323	324
325	326	327	328	329	330
331	332	333	334	335	336
337	338	339	340	341	342
343	344	345	346	347	348
349	350	351	352	353	354
355	356	357	358	359	360
361	362	363	364	365	366
367	368	369	370	371	372
373	374	375	376	377	378
379	380	381	382	383	384
385	386	387	388	389	390
391	392	393	394	395	396
397	398	399	400	401	402
403	404	405	406	407	408
409	410	411	412	413	414
415	416	417	418	419	420
421	422	423	424	425	426
427	428	429	430	431	432
433	434	435	436	437	438
439	440	441	442	443	444
445	446	447	448	449	450
451	452	453	454	455	456
457	457	459	460	461	462
463	464	465	466	467	468
469	470	471	472	473	474
475	476	477	478	479	480
481	482	483	484	485	486
487	488	489	490	491	492
493	494	495	496	497	498
499	500	501	502	503	504
505	506	507	508	509	510
511	512	513	514	515	516
517	518	519	520	521	522
523	524	525	526	527	528

My Conclusion for Real Primes in 3(mod 4):
There are only two blank squares left, the real primes 2 and 3. If I add these to my sieve algorithm, I have identified all of the primes up to 523.

Hurray!!!!!!!!!!!!!!

P.S. This is not a proof, but I'm only 10. However if somebody wrote a computer program using my sieve, I think you could make large prime numbers. Because the logic and the patterns are so strong, I predict this will continue forever. This is my hypothesis.

My sieve finds primes, but it also explains gaps between the primes. Primes congruent to 3(mod 4) also seem to appear on alternate lines in my array. These also have 'little bumps', this time of 3 and 5 (up to 528).

E.g. the real primes 383 and 419 are both congruent to 3(mod 4). Start with 383:

(A) The next line contains 385 and 389, which are congruent to 1(mod 4). Therefore already coloured in.

(B) The next line contains 391 and 395, which are congruent to 3(mod 4), BUT are 'rotten'.

Similarly, the next line follows pattern A, and the following line pattern B. Finally, the next line follows pattern A once again, and the following line contains 415 and 419. Both 415 and 419 are congruent to 3(mod 4), 415 is 'rotten', and 419 is a real prime

This explains the gap between 383 and 419. The same logic can be used to explain the other gaps.

Chapter 5: More Thoughts

I used a kind of binary to calculate the 'bumps' in the 6-column array. If there is a prime in the line, write a 1 at the end, if there isn't, write a 0. This made it very easy to identify the bumps, not just the size, but the frequency also.

I didn't consider a gap of 1 to be a 'bump', because these make up the alternate sequence: 1(mod4) is the other half to 3(mod 4) - all odds are one or the other.

In the range I investigated, I found 'bumps' of 3, 5, and 7. These could be odd numbers, or prime numbers, or only include 3, 5, 7 and repeat in different combinations. It's too early to say.

It would have taken me even longer to make the list long enough to come up with a bump of 9, however if the bumps are prime, I wouldn't expect to see 9, but 11 next. That's why it would be interesting if someone put my algorithm into a computer programme and leave it running for a long time. (A really, really long time!)

Another thought - would the bumps generate every prime number except 2??

Another thing that was interesting to me, was that in the range of 1-528 on the chart above, there were nearly the same number of bumps in each modulo: 6 'bumps' in 1(mod 4) and 7 'bumps' in 3(mod 4), although different sizes....do the frequency of bumps tally??? I would love to find this out.

I case you are wondering, yes you did read it correctly, 'rotten' numbers are different in 3 (mod 4) to 1(mod 4).

Chapter 6: Extending the Sample - to Hunt for 'Bumps'

I decided to extend my sample to look for 'bumps', but I ended up finding three very important numbers which made me amend my rule for 3(mod 4).

Consider 539, which is $7^2 \times 11$ On its own, it seems like it isn't rotten because both 7 and 11 are congruent to 3(mod 4), but $7^2 = 49$ and 49 is not congruent to 3(mod 4). This made me think about amending my rule to include powers of primes.

Similarly consider 847 which is $11^2 \times 7$ as before both 7 and 11 are congruent to 3(mod 4), but $11^2 = 121$ and 121 is not congruent to 3(mod 4).

Now consider 1463, which is not in my range, but $1463 = 7 \times 11 \times 19$, where 7, 11 and 19 are all congruent to 3(mod 4). But $7 \times 11 = 77$ which is NOT congruent to 3(mod 4). So I need to make my rule even clearer, instead of using powers of primes, I need to use multiples of primes.

*******WARNING: Additional Rule********

To cast out 'rotten numbers' in 3(mod 4), either one of the prime factors must be 'rotten' or a multiple of two of the primes is 'rotten'.

Extending my sample revealed more information, which is why it's important not to jump too early! I'm glad I did this even though it took me forever!!

Chapter 7: Proof

i) Start by defining the sequence of 1(mod 4):

$4n + 1 = x$ where n is any non-negative integer, and x is any number in the sequence. The two are interrelated.

$n + 1 = t$ where n is any non-negative integer, and t is any term in the sequence. Again, the two are interrelated.

ii) Similarly define the sequence of 3(mod 4):

$4m + 3 = y$ where m is any non-negative integer, and y is any number in the sequence. The two are interrelated.

$m + 1 = u$ where m is any non-negative integer, and u is any term in the sequence. Again, the two are interrelated.

NOTE: that the term is always one more than the origin number n or m

For example:

$$4n + 1 = x, \quad n + 1 = t \qquad \text{where } n = 5, \quad x = 21, \quad \text{and} \quad t = 6$$

And if you check, then the 6th term in 1(mod 4) is 21

iii) Follow my sieve algorithm and focus on 1(mod 4), eliminate multiples of 3

If the term is divisible by 3, then logically x is divisible by 3, so cannot be prime in the real world. Using my definition:

$n + 1 = t$ becomes $\frac{t}{3} = \frac{n+1}{3}$ so $n = 3k - 1$ where k is a non-negative integer, so substitute:

$4n + 1 = x$ where $n = 3k - 2$, $\therefore 4n = 12k - 4$, so $4n + 1 = 12k - 3$ or $4n + 1 = 3(4k - 1)$

This proves that when t is divisible by 3, x is also divisible by 3, and therefore not prime in the real world.

iv) Stay with my sieve algorithm, focus on 1(mod 4), eliminate the square composites

Square numbers in the form $(4n + 3)^2$ or $[3(\text{mod } 4)]^2$ are 'rotten'.

This because $(4n + 3)^2$ is divisible by $4n + 3$, and since $4n + 3$ is NOT congruent to 1(mod 4), numbers in this form are 'rotten'. they have a divisor that does not appear in the sequence. Therefore they must be eliminated.

v) Stay with my sieve algorithm, focus on 1(mod 4), eliminate the multiples of the square root of $(4n + 3)^2$

All multiples of the square root of $(4n + 3)^2$ or $[3(\text{mod } 4)]^2$ are 'rotten' because they are multiples of $4n + 3$

vi) Stay with my sieve algorithm, focus on 1(mod 4), eliminate the s-composites

Square numbers in the form $(4n + 1)^2$ or $[1(\text{mod } 4)]^2$ are s-composites and must be eliminated.

All multiples of the square root of $(4n + 1)^2$ or $[1(\text{mod } 4)]^2$ are s-composite because they are multiples of $4n + 1$, where BOTH of the numbers must be greater than 1.

$p \times q$ is s-composite, where both p and $q \equiv 4n + 1$ and $p > 1$ and $q > 1$

For example $5^2 = 25$ $\sqrt{25} = 5$ and $5 > 1$
$\therefore$ any multiple of 5 EXCEPT for 5 itself must be eliminated as s-composite, e.g. 65
5 cannot be eliminated because 5×1 does not fit the rule. 1 cannot be greater than 1.

THIS WILL REVEAL PRIMES CONGRUENT TO 1(MOD 4)!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!

vii) Stay with my sieve algorithm, move to 3(mod 4), eliminate the s-composites from the blank numbers that are left.

START WITH THE BLANK NUMBERS THAT ARE LEFT. This is important because it follows the pattern.

$(4n + 1) \times (4m + 3)$ is 'rotten' and must be eliminated, where $4n + 1$ has already been identified as s-prime

Next eliminate composites in the form:
$(4a+3)(4b+3)(4c+3)$ where $(4a+3)(4b+3) \equiv 1 (\text{mod } 4)$

THIS WILL REVEAL PRIMES CONGRUENT TO 3(MOD 4)!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!

I'm extremely tired, but very happy!!

<u>Chapter 8: Do the 'Bumps' Tally?</u>

I extended my sample, I wanted to find the next bump...and see if they tally.

So in the range 529 - 1056, the numbers congruent to 1(mod 4) the numbers are:

529...533...537...541...545...549...553...557...561...565...569...573...577...581...585...589...593...597...601...605...609...613...617...621...625...629...633...637...641...645...649...653...657...661...665...669...673..677...681...685..689...693...697...701...705...709...713...717...721..725...729...733...737...741...745...749...753...757...761...765...769..773...777...781...785...789...793...797...801...805...809...813...817...821...825...829...833...837...841...845...849...853...857...861...865...869...873...877...881...885...889...893...897...901...905...909...913...917...921...925..929...933...937...941...945...949...953...957...961...965...969...973...977...981...985...989...993...997...1001........1005...1009...1013...1017...1021...1025...1029...1033...1037...1041...1045...1049...1053...

Strip out the rotten numbers, according to my rule for 1(mod 4)

<u>The numbers that are left are:</u>
List of all the numbers which aren't 'rotten' are:
533...541...545...557...565...569...577...593...601....613...617...625...629...641...653...661...673..677...685..689...697...701...709..725...733...745...757...761...769..773...785...793...797...809...821...829...841...845...853...857...865...877...881...901...905...925..929...937...941...949...953...965....977...985...997...1009...1013...1021...1025...1033...1037...1049...

From this list, the s-primes are:
541...557...569...577...593...601....613...617...641...653...661...673..677...701...709...733...757...761...769...773..797...809...821...829...853...857...877...881...929...937...941...953....977...997...1009...1013...1021...1033...1049

And from this list, the s-composites are:
533...545...565...625...629...685...689...697...725...745...785...793...841...845...865...901...905...925...949...965..985...1025...1037...

Map for primes congruent to 1(mod 4)

a) PINK = rotten numbers - remove these first, following my rules
b) BLUE = s-composites left over after rotten numbers have been stripped out
c) YELLOW = s-primes congruent to 1(mod 4), stripped of 'rotten' numbers/s-composites
d) GREY = 1, multiples of 2 except 2 itself, and multiples of 3 that are not congruent to 1(mod4) except 3 itself

529	530	531	532	533	534
535	536	537	538	539	540
541	542	543	544	545	546
547	548	549	550	551	552
553	554	555	556	557	558
559	560	561	562	563	564
565	566	567	568	569	570
571	572	573	574	575	576
577	578	579	580	581	582
583	584	585	586	587	588
589	590	591	592	593	594
595	596	597	598	599	600
601	602	603	604	605	606
607	608	609	610	611	612
613	614	615	616	617	618
619	620	621	622	623	624
625	626	627	628	629	630
631	632	633	634	635	636
637	638	639	640	641	642
643	644	645	646	647	648
649	650	651	652	653	654
655	656	657	658	659	660
661	662	663	664	665	666
667	668	669	670	671	672
673	674	675	676	677	678
679	680	681	682	683	684
685	686	687	688	689	690
691	692	693	694	695	696
697	698	699	700	701	702
703	704	705	706	707	708
709	710	711	712	713	714
715	716	717	718	719	720
721	722	723	724	725	726
727	728	729	730	731	732
733	734	735	736	737	738
739	740	741	742	743	744
745	746	747	748	749	750
751	752	753	754	755	756
757	758	759	760	761	762
763	764	765	766	767	768
769	770	771	772	773	774
775	776	777	778	779	780
781	782	783	784	785	786
787	788	789	790	791	792

793	794	795	796	797	798
799	800	801	802	803	804
805	806	807	808	809	810
811	812	813	814	815	816
817	818	819	820	821	822
823	824	825	826	827	828
829	830	831	832	833	834
835	836	837	838	839	840
841	842	843	844	845	846
847	848	849	850	851	852
853	854	855	856	857	858
859	860	861	862	863	864
865	866	867	868	869	870
871	872	873	872	875	876
877	878	879	880	881	882
883	884	885	886	887	888
889	890	891	892	893	894
895	896	897	898	899	900
901	902	903	904	905	906
907	908	909	910	911	912
913	914	915	916	917	918
919	920	921	922	923	924
925	926	927	928	929	930
931	932	933	934	935	936
937	938	939	940	941	942
943	944	945	946	947	948
949	950	951	952	953	954
955	956	957	958	959	960
961	962	963	964	965	966
967	968	969	970	971	972
973	974	975	976	977	978
979	980	981	982	983	984
985	986	987	988	989	990
991	992	993	994	995	996
997	998	999	1000	1001	1002
1003	1004	1005	1006	1007	1008
1009	1010	1011	1012	1013	1014
1015	1016	1017	1018	1019	1020
1021	1022	1023	1024	1025	1026
1027	1028	1029	1030	1031	1032
1033	1034	1035	1036	1037	1038
1039	1040	1041	1042	1043	1044
1045	1046	1047	1048	1049	1050
1051	1052	1053	1054	1055	1056

The remaining 'blank' numbers in the range are all congruent to 3(mod 4). These are:
535...539...547...551...559...563...571...575...583...587...595...599...607...611...619...623...631...635...643...647...655...659...667...671...679...683...691...695...703...707...715...719...727...731...739...743...751...755...763...767...775...779...787...791...799...803...811...815...823...827...835...839...847...851...859...863...871...875...883...887...895...899...907...911...919...923...931...935...943...947...955...959...967...971...979...983...991...995...1003.....1007...1015...1019...1027...1031...1039...1043...1051...1055...

The list of all the numbers which aren't 'rotten' are:
547...563...571...587...599...607...619...631...643...647...659...683...691...719...727...739...743...751...787...811...823...827...839...859...863...883...887...907...911...919...947...967...971...983...991...1019...1031...1039...1051

Map for primes congruent to 1(mod 4) + primes congruent to 3(mod 4) = MAP FOR ALL PRIMES

e) USE the map for primes congruent to 1(mod 4) as a base
f) GREEN = rotten numbers - remove these first, following my rules
g) ORANGE = T-primes congruent to 3(mod 4), stripped of 'rotten' numbers
h) BLANK = the real primes 2 and 3

529	530	531	532	533	534
535	536	537	538	539	540
541	542	543	544	545	546
547	548	549	550	551	552
553	554	555	556	557	558
559	560	561	562	563	564
565	566	567	568	569	570
571	572	573	574	575	576
577	578	579	580	581	582
583	584	585	586	587	588
589	590	591	592	593	594
595	596	597	598	599	600
601	602	603	604	605	606
607	608	609	610	611	612
613	614	615	616	617	618
619	620	621	622	623	624
625	626	627	628	629	630
631	632	633	634	635	636
637	638	639	640	641	642
643	644	645	646	647	648
649	650	651	652	653	654
655	656	657	658	659	660
661	662	663	664	665	666
667	668	669	670	671	672
673	674	675	676	677	678
679	680	681	682	683	684
685	686	687	688	689	690
691	692	693	694	695	696
697	698	699	700	701	702
703	704	705	706	707	708
709	710	711	712	713	714
715	716	717	718	719	720
721	722	723	724	725	726
727	728	729	730	731	732
733	734	735	736	737	738
739	740	741	742	743	744
745	746	747	748	749	750
751	752	753	754	755	756
757	758	759	760	761	762
763	764	765	766	767	768
769	770	771	772	773	774
775	776	777	778	779	780
781	782	783	784	785	786
787	788	789	790	791	792

793	794	795	796	797	798
799	800	801	802	803	804
805	806	807	808	809	810
811	812	813	814	815	816
817	818	819	820	821	822
823	824	825	826	827	828
829	830	831	832	833	834
835	836	837	838	839	840
841	842	843	844	845	846
847	848	849	850	851	852
853	854	855	856	857	858
859	860	861	862	863	864
865	866	867	868	869	870
871	872	873	872	875	876
877	878	879	880	881	882
883	884	885	886	887	888
889	890	891	892	893	894
895	896	897	898	899	900
901	902	903	904	905	906
907	908	909	910	911	912
913	914	915	916	917	918
919	920	921	922	923	924
925	926	927	928	929	930
931	932	933	934	935	936
937	938	939	940	941	942
943	944	945	946	947	948
949	950	951	952	953	954
955	956	957	958	959	960
961	962	963	964	965	966
967	968	969	970	971	972
973	974	975	976	977	978
979	980	981	982	983	984
985	986	987	988	989	990
991	992	993	994	995	996
997	998	999	1000	1001	1002
1003	1004	1005	1006	1007	1008
1009	1010	1011	1012	1013	1014
1015	1016	1017	1018	1019	1020
1021	1022	1023	1024	1025	1026
1027	1028	1029	1030	1031	1032
1033	1034	1035	1036	1037	1038
1039	1040	1041	1042	1043	1044
1045	1046	1047	1048	1049	1050
1051	1052	1053	1054	1055	1056

I wanted to find the next sized 'bump', but the largest gap remained 7 in my range. I won't solve this until I can write a computer programme, and it is irritating to be 10. Arggghhhh!!!!

But the frequency of gaps did stay in roughly the same. And I am interested in this, I think it's a good question to consider. In the sample I looked at there were 16 gaps in 1(mod 4), and 17 gaps in 3(mod 4).

That's a lot of work for one sentence!

Final Thoughts:

I'm really glad I did this, I leant a lot, just by thinking. Numbers are beautiful.